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Letter to the Editor

Chebyshev polynomial approximation for dynamical response problem of random system

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1. Introduction

In reality, a physical system often includes some uncertain parameters. These uncertain parameters usually can be modelled as random variables with known statistical properties. A system with random variables as its coefficients is known as a random system. Thus, there is a need for statistical analysis of the response problem of a random system. The dynamical response problems of a random system subject to random excitations are very complicated in nature, especially for evolutionary random response problems. Mainly there are three kinds of mathematical methods available for dealing with these complicated response problems. The first one is the Monte-Carlo method [1], which is simple and universal, but usually involved with a quite amount of computational effort. The second one is the stochastic perturbation method, which is involved with the least computational effort, but usually restricted to systems with random variables of small fluctuations only [2]. The third one is the orthogonal polynomial approximation method, which is free from the small perturbation assumption, thus providing more applicability, and which is involved with moderate computational effort, but with a quite amount of mathematical deductions [3–5]. Moreover, the choice of the orthogonal basis depends upon the probability density functions (PDF) of the random variables. Two typical probability density functions of random parameters are commonly chosen in response analysis of random systems, namely, the normal distribution and the uniform distribution in a finite interval. In this regard, Hermite Polynomials are chosen for the former one and Legendre Polynomials are chosen for the latter one as the orthogonal basis respectively. However, taking the normal distribution runs the risk of that part of the sample systems may have negative parameters, which would result in instability of these sample systems. Taking the uniform distribution assumption for random variables varying between –1 and +1, will not risk the instability problem. For an alternative choice, an arch-like probability density function may be also reasonable in reality, and will not

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take the risk of instability, too. An arch-like probability density function in Fig. 1 can be described as follows:

$$p(\xi) = \begin{cases} (2/\pi)\sqrt{1-\xi^2}, & \text{when } |\xi| \leqslant 1, \\ 0, & \text{when } |\xi| > 1. \end{cases} \tag{1}$$

In accordance with this PDF, Chebyshev Polynomials of the second kind can be chosen as the orthogonal basis.

2. Chebyshev Polynomials

The general expression for Chebyshev Polynomials of the second kind can be put as follows [6]:

$$U_n(\xi) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!(n-2k)!} (2\xi)^{n-2k}. \tag{2}$$

Thus, we have

$$\begin{aligned} U_0(\xi) &= 1, \\ U_1(\xi) &= 2\xi, \\ U_2(\xi) &= 4\xi^2 - 1, \\ U_3(\xi) &= 8\xi^3 - 4\xi, \\ U_4(\xi) &= 16\xi^4 - 12\xi^2 + 1, \\ U_5(\xi) &= 32\xi^5 - 32\xi^3 + 6\xi. \end{aligned} \tag{3}$$

The orthogonality of Chebyshev Polynomials of the second kind can be expressed as

$$\int_{-1}^{1} \frac{2}{\pi}\sqrt{1-\xi^2}\, U_i(\xi) U_j(\xi)\, \mathrm{d}\xi = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4}$$

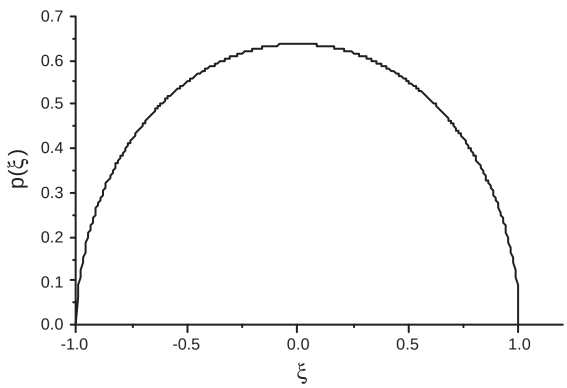

Fig. 1. The arch-like PDF curve for ξ.

The recurrent formula for Chebyshev Polynomials of the second kind is

$$U_{n+1}(\xi) = 2\xi U_n(\xi) - U_{n-1}(\xi), \quad n = 1, 2\ldots; \quad U_1(\xi) = 2\xi U_0(\xi). \tag{5}$$

Eq. (4) represents a weighted orthogonal relationship. Since the weighting function is just the same as the arch-like PDF, $p(\xi)$ in Eq. (1), the left-hand side of Eq. (4) may be regarded as the expectation of the product $U_i(\xi)U_j(\xi)$. It is well known that owing to the orthogonality of Chebyshev Polynomials, any measurable function $f(\xi)$ can be expressed into the following series form:

$$f(\xi) = \sum_{i=0}^{\infty} \gamma_i U_i(\xi), \tag{6}$$

where

$$\gamma_i = \int_{-1}^{1} p(\xi) f(\xi) U_i(\xi)\, d\xi. \tag{7}$$

Similar results can be obtained for measurable functions of several mutually independent random variables. For example, suppose ξ_1 and ξ_2 are two mutually independent random variables with the same PDF as in Eq. (1), noted as $p(\xi_1)$ and $p(\xi_2)$ respectively; $U_i(\xi_1)$ and $U_j(\xi_2)$ are Chebyshev Polynomials of the second kind for ξ_1 and ξ_2, respectively, then any two-dimensional measurable function $f(\xi_1, \xi_2)$ can be expressed into a double series form

$$f(\xi_1, \xi_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \gamma_{ij} U_i(\xi_1) U_j(\xi_2), \tag{8}$$

where

$$\gamma_{ij} = \int_{-1}^{1} \int_{-1}^{1} p(\xi_1) p(\xi_2) f(\xi_1, \xi_2) U_i(\xi_1) U_j(\xi_2)\, d\xi_1\, d\xi_2. \tag{9}$$

It is noted that Eqs. (6) and (8) are valid only for taking the sum over infinite number of items. If in practice only a limited number of items are taken in Eqs. (6) or (8), then either of the results is merely an approximation with a minimal mean square residual.

3. Dynamical response analysis of random system

The discrete model of a random structure usually takes a multi-degree-of-freedom linear system, of which the random differential equation can be expressed as

$$M\ddot{x} + C\dot{x} + Kx = F(t), \tag{10}$$

where M, C, and K are mass, damping and stiffness matrices, respectively, all with random variable elements of given statistical properties, and the sample matrices of M are supposed to be positive-definite; x is an n-vector response; and $F(t)$ is an n-vector excitation. Thus, the response x must be a random one, whether the excitation $F(t)$ is deterministic or not. On the other hand, if the excitation $F(t)$ is an evolutionary random process, so is the response, whether the system is random or not. A unified approach to evolutionary random response problems was suggested in Refs. [7,8] for a deterministic system, whether it is time-independent or not. This unified approach can also be applied to random systems in the following two ways. Since any sample of a random system is a deterministic one, one way is to apply the unified approach directly to the evolutionary random response problem of a sample system. Then, the ensemble response characteristics are obtained through the Monte-Carlo simulation for the random system parameters only. The other way is to apply either the stochastic perturbation technique or the orthogonal approximation method to reduce the random system into its deterministic equivalent. Then, the evolutionary random response problem of the equivalent deterministic system is solved by the unified approach. As a matter of fact, the evolutionary random response problem under the Niigata earthquake excitation was solved for a random shear beam by the first way [9], and for a discrete structure model by the second way via the perturbation analysis [10]. Now let us look for how to reduce the random system into its deterministic equivalent by the Chebyshev Polynomial approximation.

For simplicity and readability, let M and C in Eq. (10) be deterministic matrices, and K the random one only. And suppose that all the elements in K only depend upon two mutually independent physical random stiffness coefficients, ξ_1 and ξ_2, both with an arch-like PDF described by Eq. (1). Then, the random stiffness matrix K can be expressed as a sum of its mean matrix $\bar{K}$ and two other matrices, each proportional to one of the independent physical random stiffness coefficients, namely

$$K = \bar{K} + \sum_{i=1}^{2} \xi_i K_i, \tag{11}$$

where $\bar{K}$ and K_i are deterministic matrices.

In general, the response vector of system (10) should be a 3-dimensional function $x(t, \xi_1, \xi_2)$. However, under the assumption that the statistical properties of the random system parameters are independent of the statistical properties of the random excitation, the response x can be separable in time and the random variables, so that it is possible to look for the response in the form

$$x(t, \xi_1, \xi_2) = X(t) f(\xi_1, \xi_2),$$

where $f(\xi_1, \xi_2)$ is just a scalar function of ξ_1 and ξ_2. Now by Chebyshev Polynomials, $U_i(\xi_1)$ and $U_j(\xi_2)$, $i = 0, 1, 2, \ldots, n_1; j = 0, 1, 2, \ldots, n_2$, $f(\xi_1, \xi_2)$ can be further expanded approximately into a double series of finite terms of orthogonal polynomials. So it is reasonable to look for the system response approximately in the following form:

$$x(t, \xi_1, \xi_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} X_{ij}(t) U_i(\xi_1) U_j(\xi_2), \tag{12}$$

where the subscript i runs for sequential number of Chebyshev Polynomials of ξ_1, and j runs for that of ξ_2.

By substituting Eqs. (11) and (12) into Eq. (10), we have

$$\left(M \frac{\mathrm{d}^2}{\mathrm{d}t^2} + C \frac{\mathrm{d}}{\mathrm{d}t} + \bar{K} + K_1 \xi_1 + K_2 \xi_2 \right) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} X_{ij}(t) U_i(\xi_1) U_j(\xi_2) = F(t).$$

Note that the recurrent formula, Eq. (5), can be rewritten as

$$\xi U_n(\xi) = \tfrac{1}{2}\{U_{n+1}(\xi) + U_{n-1}(\xi)\}.$$

Hence, we have

$$\begin{aligned}&\left(M\frac{d^2}{dt^2} + C\frac{d}{dt} + \bar{K}\right)\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} U_i(\xi_1)U_j(\xi_2)X_{ij}(t)\\&+\frac{1}{2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}[K_1\{U_{i+1}(\xi_1) + U_{i-1}(\xi_1)\}U_j(\xi_2) + K_2 U_i(\xi_1)\{U_{j+1}(\xi_2) + U_{j-1}(\xi_2)\}]X_{ij}(t)\\&= F(t).\end{aligned} \tag{13}$$

Multiplying both sides of Eq. (13) by $U_i(\xi_1)U_j(\xi_2)$ in sequence, and then taking the expectations, owing to the orthogonality relationships of the Chebyshev Polynomials, we finally obtain the following equivalent deterministic system of Eq. (13)

$$\begin{aligned}&\left(M\frac{d^2}{dt^2} + C\frac{d}{dt} + \bar{K}\right)X_{ij}(t) + \tfrac{1}{2}[K_1\{X_{i+1,j}(t) + X_{i-1,j}(t)\} + K_2\{X_{i,j+1}(t) + X_{i,j-1}(t)\}]\\&\quad= F(t)\delta_{0i}\delta_{0j}, \quad i = 0, 1, 2, \ldots, n_1, \quad j = 0, 1, 2, \ldots, n_2,\end{aligned} \tag{14}$$

where $X_{-1,j}(t), X_{i,-1}(t), X_{n_1+1,j}(t)$, and $X_{i,n_2+1}(t)$ are supposed to be zero. If we take the permutation of the subscript ij of $X_{ij}(t)$ sequentially in such a way that the sums of $i+j$ are arranged in non-decreasing order, then the stiffness matrix of Eq. (14) will be a sparse or band one with matrix elements of which the order is the same as the stiffness matrix in the original system (10).

Now in Eq. (14) the system itself is deterministic, any effective method for response problems of deterministic system can be applied to it, especially the unified approach for evolutionary random response problems can be applied to it as well.

Owing to the assumption of $x(t, \xi_1, \xi_2)$, Eq. (12), and the orthogonality relationship, Eq. (4), once the covariance matrices of all the $X_{ij}(t)$ are obtained, the covariance matrix of $x(t)$ can be obtained as

$$E[x(t)x^{T}(t)] = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} E[X_{ij}(t)X_{ij}^{T}(t)]. \tag{15}$$

In our presentation essential for derivation are the following two assumptions: (1) the mutual independence of random parameters, (2) the independence between random parameters and random excitations. For the sake of clarity and readability we have just discussed the dynamic response problems of structures with a random stiffness matrix only, but the method applies equally well in principle to those structures with a random stiffness matrix, together with a random mass matrix and a random damping matrix, as long as these matrices satisfy the above two assumptions.

4. Two numerical examples

4.1. Earthquake response of a SDOF random system

Consider a single-degree-of-freedom random system subject to the 1964 Niigata earthquake. The evolutionary spectrum of the ground acceleration $w(t)$ may be expressed as

$$S_w(t,\omega) = |A(t,\omega)|^2 S_f(\omega), \tag{16}$$

where $S_f(\omega) = S_0 = 2\,\mathrm{cm}^2/\mathrm{s}^3$, and

$$A(t,\omega) = \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{\max(\mathrm{e}^{-at} - \mathrm{e}^{-bt})}\left\{\frac{\Omega^4(t) + 4\varsigma^2(t)\Omega^2(t)\omega^2}{[\omega^2 - \Omega^2]^2 + 4\varsigma^2(t)\Omega^2(t)\omega^2}\right\}^{1/2} \tag{17}$$

with $a = 0.25\,\mathrm{s}^{-1}$, $b = 0.5\,\mathrm{s}^{-1}$, and

$$\varsigma(t) = \begin{cases} 0.64, & 0 \leqslant t \leqslant 4.5\,\mathrm{s}, \\ 1.25(t-4.5)^3 - 1.875(t-4.5)^2 + 0.64, & 4.5\,\mathrm{s} \leqslant t \leqslant 5.5\,\mathrm{s}, \\ 0.015, & t \geqslant 5.5\,\mathrm{s}, \end{cases}$$

$$\Omega(t) = \begin{cases} 15.56\,\mathrm{rad/s}, & 0 \leqslant t \leqslant 4.5\,\mathrm{s}, \\ 27.12(t-4.5)^3 - 40.68(t-4.5)^2 + 15.56\,\mathrm{rad/s}, & 4.5\,\mathrm{s} \leqslant t \leqslant 5.5\,\mathrm{s}, \\ 2.0\,\mathrm{rad/s}, & t \geqslant 5.5\,\mathrm{s}. \end{cases}$$

The random differential equation of the relative motion of the structure to the ground may be written as

$$m\ddot{x} + c\dot{x} + kx = -mw(t),$$

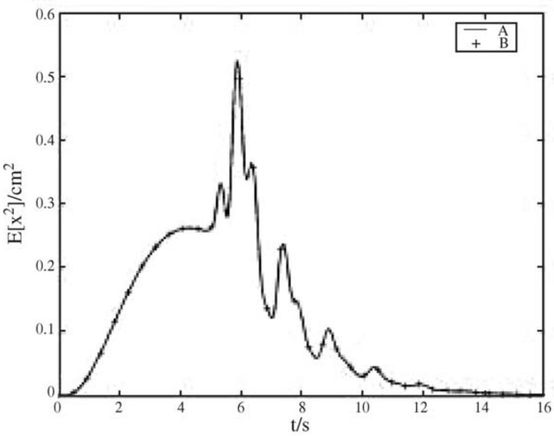

Fig. 2. Mean square random responses of x. (A) for Chebyshev polynomial approximation, (B) for Monte-Carlo simulation.

where $m = 1 \times 10^3$ kg, $c = 6.3 \times 10^2$ N s/m, k is a random parameter, which can be expressed as $k = \bar{k} + \xi k_1$, with $\bar{k} = 39.48 \times 10^3$ N/m, $k_1 = 3.95$ N/m, and ξ is a physical random stiffness coefficient with an arch-like PDF described by Eq. (1).

In calculation we took U_0 up to U_4 only. The numerical results for $E[x^2]$ are shown in Fig. 2, where curve A is the numerical result by the Chebyshev Polynomial approximation, and curve B results from the Monte-Carlo simulation method. Curves A and B almost coincide.

In our simulations the Monte-Carlo method is applied to samples of random parameters only. The mean square evolutionary random responses for the sample systems are obtained by the unified approach. Thus, the simulation efforts are greatly reduced as compared with those for directly applying the Monte-Carlo method to combinations of both samples of random parameters and random excitations. In this example the number of simulation samples for ξ is taken as 50, equally spaced in -1 to $+1$.

4.2. Earthquake response of a three-story random structure

Consider a three-story random structure model subject to the same earthquake excitation as in the previous example (Fig. 3). The random differential equation of the relative motion of the structure to the ground may be written as

$$M\ddot{x} + C\dot{x} + Kx = gw(t),$$

where

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}, \quad g = -\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}.$$

We assume that matrices M and C are deterministic, and K is random with a deterministic constant parameter k_1 and two random parameters k_2 and k_3. Suppose that K can be

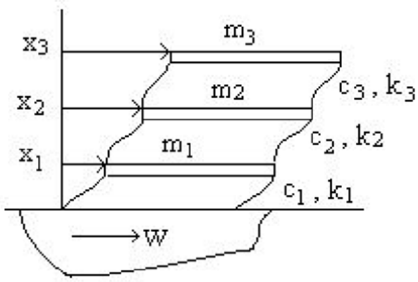

Fig. 3. A 3-story random structure model.

expressed as

$$K = \bar{K} + \sum_{i=1}^{2} \xi_i K_i,$$

where ξ_1 and ξ_2 are physical random stiffness coefficients with the same arch-like PDF described by Eq. (1). And the data are taken as follows:

$$m_1 = m_2 = m_3 = 2.917 \times 10^4\ \text{kg}, \quad c_1 = c_2 = c_3 = 2.5 \times 10^5\ \text{N s/m},$$

$$\bar{K} = 3.5 \times 10^7 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{N/m},$$

$$K_1 = 3.5 \times 10^6 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{N/m},$$

$$K_2 = 7.0 \times 10^6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{N/m}.$$

We took $n_1 = n_2 = 3$ in this example. The numerical results for the mean square evolutionary random responses of the top floor, $E[x_3^2]$, are shown in Fig. 4, where curve A results from the three-story random structure by the Chebyshev Polynomial approximation; curve B results from the sample structure with the maximum k_2 and k_3; and curve C results from the sample structure with the minimum k_2 and k_3.

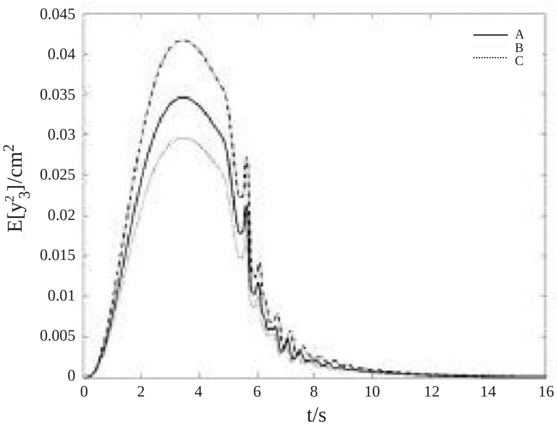

Fig. 4. Mean square random responses of the top floor. (A) for the random system, (B) for the sample system with maximum k_2 and k_3 (C) for the sample system with minimum k_2 and k_3.

5. Conclusions

An alternative PDF for random parameters of a random system, namely an arch-like PDF is suggested in this paper. To match such a PDF, a Chebyshev Polynomial approximation for reducing the random system into its deterministic equivalent is also presented. Thus the response problem of such a kind of random system can be transformed into that of a deterministic system, so that any available effective method for solving the dynamic response problem of a deterministic system can be applied to it. Particularly the unified approach to evolutionary random response problems for a deterministic system now can be applied to a random system as well. Numerical examples show that the suggested method is effective. As a matter of fact, the suggested Chebyshev Polynomial approximation can be viewed as a new variation of the weighted residual method in random space, so are the similar methods by other forms of orthogonal polynomial approximations.

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